M.A./M.Sc (First Semester) Examination,
2015-16
MATHEMATICS
Paper-Third Topology-I

MODEL Answer (AV-8205)
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1. (i) Let X be any set. The collection $I = \{X, \emptyset\}$ consisting of the empty set and the whole space is always a topology for X called the indiscrete topology. The pair (X, I) is called an indiscrete topological space.

~~(iii)~~ Let D be the collection of all subsets of X, then D is a topology for X called the discrete topology. The pair (X, D) is called a discrete topological space.

(ii) example $\mathfrak{I}_1 = \{X, \{a\}, \emptyset\}$, $\mathfrak{I}_2 = \{X, \emptyset, \{b\}\}$ and $\mathfrak{I}_3 = \{X, \emptyset, \{c\}\}$ or any three also consider

(iii) Let (R, U) be the usual topological space and let $F_n = [\frac{1}{n}, 1]$, $n \in N$ so that F_n is a closed interval of R.

F_n is a U-closed set. Now

$$\cup \{F_n : n \in N\} = \{1\} \cup [\tfrac{1}{2}, 1] \cup [\tfrac{1}{4}, 1] \ldots = (0, 1]$$

Since $(0,1]$ is not closed.

(iv) Findin $N(b)$ — 1M and Findin $N(c)$ — 1M

(v) Let $G \in \mathfrak{I}$ and $x \in G$. Since G is $\mathfrak{I}$-open, it is $\mathfrak{I}$-nbd of x and since $\mathcal{B}$ is a base for $\mathfrak{I}$, there exists $B \in \mathcal{B}$, such that $x \in B \subset G$. Since $\mathcal{B}$ is a base for $\mathfrak{I}'$ and $B \in \mathcal{B}$ such that $B \in \mathfrak{I}'$. Hence G is $\mathfrak{I}'$-nbd of x. Since x is arbitrary, $G \in \mathfrak{I}'$. Thus $\mathfrak{I} \subset \mathfrak{I}'$, by symmetry $\mathfrak{I}' \subset \mathfrak{I}$ implies that $\mathfrak{I} = \mathfrak{I}'$.

(vi) Pasting Lemma: Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f and g combine to give a continuous function $h : X \to Y$, defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

(vii) Let $(X, \mathfrak{J})$ and $(Y, \mathfrak{J}')$ be two topological space. valid definition of uniform continuity in topological space is considered.

(viii) A Second countability axiom: A topological space $(X, \mathfrak{J})$ is said to be second countable iff there exists a countable base for $\mathfrak{J}$. And also First countability axiom def also needed

(ix) Let (X, d) be a metric space with metric topology $\mathfrak{J}$. Let $x \in X$ and G be an open set containing x. By definition of an open set in a metric space, there exists a positive number r such that $S(x, r) \subset G$.
Evidently $S(x, \frac{1}{2}r) = S[x, \frac{1}{2}r] \subset G$.
This leads that to each nbd G of x, there exists a spherical nbd $S(x, \frac{1}{2}r)$ of x such that $\bar{S}(x, \frac{1}{2}r) \subset G$. Then X is regular

(x) Let $X = \{a, b, c\}$ and let $\mathfrak{J} = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$.
$\mathfrak{J}$-closed subsets of X are $X, \{b, c\}, \{a, c\}, \{c\}, \emptyset$.
Here no pair of non-empty closed sets is disjoint. Hence the only pairs which can be disjoint are $\emptyset$, A where A is any non-empty set. Now $\emptyset$ and X are $\mathfrak{J}$-open set such that $\emptyset \subset \emptyset, A \subset X, \emptyset \cap X = \emptyset$.
It follows that the space $(X, \mathfrak{J})$ is normal. But this space is not regular space since $\{b, c\}$ is closed set and $a \notin \{b, c\}$ but there donot exist disjoint open sets containing a and $\{b, c\}$

2 (a) Here $\{\mathcal{T}_\lambda : \lambda \in \Lambda\}$ is a collection of a topologies on X.

claim: $\cap\{\mathcal{T}_\lambda : \lambda \in \Lambda\}$ is also a topology on X. If $\Lambda = \phi$, then $\cap\{\mathcal{T}_\lambda : \lambda \in \Lambda\} = P(X)$.

In this case the intersection of topologies is the discrete topologies

Now, let $\Lambda \neq \phi$

[T1] : Since $\mathcal{T}_\lambda$ is a topology $\forall \lambda \in \Lambda$. it follows that $\phi, X \in \mathcal{T}_\lambda$ $\forall \lambda \in \Lambda$. But $\phi \in \mathcal{T}_\lambda \forall \lambda \in \Lambda \Rightarrow \phi \in \cap\{\mathcal{T}_\lambda : \lambda \in \Lambda\}$

and $X \in \mathcal{T}_\lambda \forall \lambda \in \Lambda \Rightarrow X \in \cap\{\mathcal{T}_\lambda : \lambda \in \Lambda\}$

[T2] : Let $G_1, G_2 \in \cap\{\mathcal{T}_\lambda : \lambda \in \Lambda\}$.

Then $G_1, G_2 \in \mathcal{T}_\lambda \forall \lambda \in \Lambda$. Since $\mathcal{T}_\lambda$ is a topology for X. it follows that $G_1 \cap G_2 \in \mathcal{T}_\lambda \forall \lambda \in \Lambda$

Hence $G_1 \cap G_2 \in \cap\{\mathcal{T}_\lambda : \lambda \in \Lambda\}$

[T3] : Let $G_\alpha \in \cap\{\mathcal{T}_\lambda : \lambda \in \Lambda\}$ for $\alpha \in \Lambda$, where Λ is an arbitrary set. Then $G_\alpha \in \mathcal{T}_\lambda \forall \lambda \in \Lambda$ and $\alpha \in \Lambda$.

Since each $\mathcal{T}_\lambda$ is a topology for X, it follows that $\cup\{G_\alpha : \alpha \in \Lambda\} \in \mathcal{T}_\lambda \forall \lambda \in \Lambda$. Hence $\cup\{G_\alpha : \alpha \in \Lambda\}$ is $\cap\{\mathcal{T}_\lambda : \lambda \in \Lambda\}$. Thus $\cap\{\mathcal{T}_\lambda : \lambda \in \Lambda\}$ is a topology for X.

(b) Since X is a $\mathcal{T}$-open set, it is a nbd of each of its points. Hence by definition of base, for every $x \in X$ there exists some $B \in \mathcal{B}$ such that $x \in B \subset X$. i.e. $X = \cup\{B : B \in \mathcal{B}\}$

the second condition: If $B_1 \in \mathcal{B}$ and $B_2 \in \mathcal{B}$ then B_1 and B_2 are $\mathcal{T}$-open. Hence their intersection $B_1 \cap B_2$ is also $\mathcal{T}$-open and therefore $B_1 \cap B_2$ is a nbd of each of its points and so by def, to each $x \in B_1 \cap B_2$, there exists $B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$

3 (a) Since $A \subset Y$, we have $A = A \cap Y$ so that A is the intersection of Y with a set open in X. But we know that in a topological space any subset in X (say Y), then its the collection $G_\lambda = \{G \cap Y : G \in \mathfrak{J}\}$ is a topology on Y. and also apply the following condition: a subset A of Y is closed in Y iff there exists a set F closed in X such that $A = F \cap Y$. Apply the above results then we conclude our result.

i.e It follows that our result follows.

(b) Let X and Y be two topological spaces and $f : X \to Y$ a continuous map. Let $\langle x_n \rangle$ be any ~~seq.~~ in X converging to a point $x_0 \in X$. We wish to show that the ~~seq.~~ $\langle f x_n \rangle$ in Y converges to $f(x_0) \in Y$. Let H be any open set in Y such that ~~the sequence~~ $f x_0 \in H$. i.e $x_0 \in f^{-1}(H)$. Since f is continuous. $f^{-1}(H)$ is an open set in X containing x_0. Since $x_n \to x_0$. $f^{-1}(H)$ contains all except a finite number of terms of the ~~seq.~~ $\langle x_n \rangle$.

Now. $x_n \in f^{-1}(H) \Rightarrow f x_n \in f[f^{-1}(H)] \doteq H$.

Thus $f(x_n) \in H$ for almost all $n \in \mathbb{N}$ and therefore $f x_n \to f x_0$.

Hence f is sequentially continuous.

4 Let A be compact relative to X and let $\{V_\lambda : \lambda \in \Lambda\}$ be a collection of sets, open relative to Y, which covers A so that $A \subset \cup\{V_\lambda : \lambda \in \Lambda\}$. Then there exists G_λ, open relative to X, such that $V_\lambda = Y \cap G_\lambda$ for every $\lambda \in \Lambda$. It follows that

$$A \subset \cup\{G_\lambda : \lambda \in \Lambda\}$$

so that $\{G_\lambda : \lambda \in \Lambda\}$ is an open cover of A relative to X. Since A is compact relative to X there exists finitely many indices $\lambda_1, \lambda_2, \ldots \lambda_n$ such that

$$A \subset G_{\lambda_1} \cup G_{\lambda_2} \cup \cdots \cup G_{\lambda_n}.$$

Since $A \subset Y$, we have

$$A \subset Y \cap [G_{\lambda_1} \cup G_{\lambda_2} \cup \cdots \cup G_{\lambda_n}] = (Y \cap G_{\lambda_1}) \cup \cdots \cup (Y \cap G_{\lambda_n}).$$

Since $Y \cap G_{\lambda_i} = V_{\lambda_i}$ $(i = 1, 2, \ldots n)$ we obtain

$$A \subset V_{\lambda_1} \cup \cdots \cup V_{\lambda_n}$$

This shows that A is compact relative to Y.

Conversely: Let A be compact relative to Y and let $\{G_\lambda : \lambda \in \Lambda\}$ be a collection of open subsets of X which cover A so that $A \subset \cup\{G_\lambda : \lambda \in \Lambda\}$. —— ①

Since $A \subset Y$, ① implies that

$$A \subset Y \cap [\cup\{G_{\lambda_i} : \lambda \in \Lambda\}] = \cup\{Y \cap G_\lambda : \lambda \in \Lambda\}.$$

Since $Y \cap G_\lambda$ is open relative to Y, the collection

$$(Y \cap G_\lambda : \lambda \in \Lambda)$$

is an open cover of A relative to Y. Since A is compact relative to Y. we must have

$$A \subset (Y \cap G_{\lambda_1}) \cup \cdots \cup (Y \cap G_{\lambda_n}) \quad \text{—— ②}$$

for some choice of finitely many indices $\lambda_1, \ldots \lambda_n$. But ② implies that $A \subset G_{\lambda_1} \cup \cdots \cup G_{\lambda_n}$.

I follows that A is compact relative to X

5 (a) Let x be any arbitrary point of X. Since $(X, \mathcal{J})$ is first countable, there exists a countable local base

$$B(x) = \{B_n : n \in \Lambda \subset \mathbb{N}\} \text{ at } x$$

Let $D_1 = B_1, D_2 = B_1 \cap B_2, \ldots, D_n = B_1 \cap B_2 \cap \ldots \cap B_n, \ldots$

Then $D_1 \supset D_2 \supset D_3 \supset \ldots \supset D_n$ and each D_n is a nbd of x.

Now, let M be any nbd of x. Since $B(x)$ is a local base at x, there exists a positive integer n_0 such that $B_{n_0} \subset M$ and since $D_{n_0} \subset B_{n_0}$, we have $D_{n_0} \subset M$. It follows that

$$D(x) = \{D_n : n \in \Lambda \subset \mathbb{N}\}.$$

is a nested local bases at x.

(b) Let Y be a closed subset of a Lindelof space $(X, \mathcal{J})$. We wish to show that the subspace $(Y, \mathcal{J}_Y)$ is Lindelof. Let $\{V_\lambda\}$ be any $\mathcal{J}$-open cover of Y. Then there exists $\mathcal{J}$-open sets G_λ such that $V_\lambda = G_\lambda \cap Y$ so that $\{G_\lambda\}$ is a $\mathcal{J}$-open cover of Y. The family consisting of all G_λ's and $X - Y$ is then a $\mathcal{J}$-open cover of X. Since X is Lindelof, this cover has a countable subcover. If we delete this from subcover the set $X - Y$, we obtain countable subcover of Y. Let this subcover be

$$\{G_{\lambda_n} : n \in \Lambda \subset \mathbb{N}\}.$$

But then $\{V_\lambda : n \in \Lambda \subset \mathbb{N}\}$ is a countable subcover of the cover $\{V_\lambda\}$ in the subspace $(Y, \mathcal{J}_Y)$. Hence the subspace $(Y, \mathcal{J}_Y)$ is Lindelof

6 (a) Let x_1, x_2 be any two distinct points of X. Since f is one-one, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

Let $y_1 = fx_1$, $y_2 = fx_2$ so that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$.

Then $y_1, y_2 \in Y$ such that $y_1 \neq y_2$. Since $(Y, \mathcal{J}')$ is H-space, there exists $\mathcal{J}'$-open sets H_1 and H_2 such that $y_1 \in H_1$, $y_2 \in H_2$ and $H_1 \cap H_2 = \emptyset$. Since f is continuous $f^{-1}(H_1)$ and $f^{-1}(H_2)$ are $\mathcal{J}$-open. Now

$$f^{-1}(H_1) \cap f^{-1}(H_2) = f^{-1}(H_1 \cap H_2) = f^{-1}(\emptyset) = \emptyset$$

and $\quad y_1 \in H_1 \Rightarrow f^{-1}(y_1) \in f^{-1}(H_1) \Rightarrow x_1 \in f^{-1}(H_1)$

$\quad y_2 \in H_2 \Rightarrow f^{-1}(y_2) \in f^{-1}(H_2) \Rightarrow x_2 \in f^{-1}(H_2)$.

Thus it is shown that for every pair of distinct points x_1, x_2 of X, there exists disjoint $\mathcal{J}$-open sets $f^{-1}(H_1)$ and $f^{-1}(H_2)$ such that $x_1 \in f^{-1}(H_1)$ and $x_2 \in f^{-1}(H_2)$.

Therefor $(X, \mathcal{J})$ is a H-space

(b) Let x be any point of X and let N be any $\mathcal{J}$-nbd of x. By hypothesis, there exists a $\mathcal{J}$-closed nbd M of x which is a regular sub-space of X. Let $\mathcal{J}_M$ denote the relative topology for M. Then $N \cap M$ is a $\mathcal{J}_M$-nbd of x. Since $(M, \mathcal{J}_M)$ is a regular space, by the preceeding theorem there exists $\mathcal{J}_M$-closed nbd W of x such that

$$W \subset N \cap M \subset N.$$

Also since W is $\mathcal{J}_M$-closed. There exists a $\mathcal{J}$-closed set V such that $\quad W = V \cap M$.

Again since V and M are $\mathcal{J}$-closed sets, it follows that $V \cap M = W$ is also $\mathcal{J}$-closed. Thus W is also a $\mathcal{J}$-closed nbd of x such that $W \subset N$. Hence $(X, \mathcal{J})$ is a regular space

7 (a) Let $(X, \mathfrak{J})$ be a regular space and let $(Y, \mathfrak{J}')$ be a homeomorphism image of $(X, \mathfrak{J})$ under a homeomorphism f.

To show that $(Y, \mathfrak{J}')$ is also a regular space. Let F be a $\mathfrak{J}'$-closed subset of Y and let q be a point of Y such that $q \notin F$.

Since f is a one-one onto map, there exists $p \in X$ such that

$$f(p) = q \iff f^{-1}(q) = p.$$

Again since f is $\mathfrak{J}$-$\mathfrak{J}'$ continuous. $f^{-1}(F)$ is $\mathfrak{J}$-closed.

Also $\quad q \notin F \Rightarrow f^{-1}(q) \notin f^{-1}(F) \Rightarrow p \notin f^{-1}(F)$.

Thus $f^{-1}(F)$ is a $\mathfrak{J}$-closed set and p is a point of X such that $p \notin f^{-1}(F)$. Since $(X, \mathfrak{J})$ is a regular space there exists $\mathfrak{J}$-open sets G and H such that

$$p \in G, \; f^{-1}(F) \subset H \text{ and } G \cap H = \emptyset.$$

Now $\quad f^{-1}(F) \subset H \Rightarrow f[f^{-1}(F)] \subset f(H) \Rightarrow F \subset f(H)$.

and $\quad G \cap H = \emptyset \Rightarrow f(G \cap H) = f(\emptyset) \Rightarrow f(G) \cap f(H) = \emptyset$.

Also since f is an open map ~~for~~ $G_1 = f(G)$ and $H_1 = f(H)$ are $\mathfrak{J}'$-open sets. Thus there exist $\mathfrak{J}'$-open sets G_1 and H_1 such that $q \in G_1$, $F \subset H_1$ and $G_1 \cap H_1 = \emptyset$.

It follows that $(Y, \mathfrak{J}')$ is also a regular space.

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7(b) Let X be a T_4-space. Then by definition, it is a normal T_1-space. To show that $(X, \mathfrak{J})$ is a Tychonoff space. It suffices to show that the space is completely regular. Let F be a $\mathfrak{J}$-closed subset of X and let x be a point of X such that $x \notin F$. Since the space is T_1. $\{x\}$ is a closed subset of X, Thus F and $\{x\}$ are disjoint closed subsets of X. Again since the space is normal. by ~~Urs~~ Urysohn's lemma there exists a continuous mapping $f: X \to [0,1]$ such that $f(\{x\}) = \{0\}$ i.e. $f(x) = 0$ and $f(F) = \{1\}$. It follows that the space $(X, \mathfrak{J})$ is completely regular

8 Urysohn's Lemma: Let F_1, F_2 be any pair of disjoint closed sets in a normal space X. Then there exists a continuous map $f: X \to [0,1]$ such that $f(x) = 0$ for $x \in F_1$ and $f(x) = 1$ for $x \in F_2$.

Proof: Let D be the set of all dyadic ~~for~~ $(0,1)$ then there exists a collection $\{G_d : d \in D\}$ of $\mathfrak{J}$-open subsets of X such that

$$d_1 < d_2 \Rightarrow F_1 \subset G_{d_1} \subset \bar{G}_{d_1} \subset G_{d_2} \subset \bar{G}_{d_2} \subset X - F_2 \quad \forall\, d_1 d_2 \in D.$$

Now we define a mapping f of X into $\mathbb{R}$ such that

$$f(x) = 0 \text{ is } x \in G_d \text{ for every } d \in D$$
$$f(x) = \sup\{d \in G_d : x \in X - G_d\} \text{ otherwise.}$$

Now if $x \in F_1$ than $x \in G_d$ for every $d \in D$ hence by definition $f(x) = 0$. Thus $f(x) = 0 \ \forall x \in F_1$.

Again if $x \in F_2$ then $x \in X - G_d$ for every $d \in D$ hence by definition $f(x) = \begin{cases} \sup(D) = 1 & \forall x \notin F_2 \\ 1 & \forall x \in F_2. \end{cases}$

Now $f(x)$ is either 0 or sup of a non-empty subset of D and $D \subset (0,1)$.

It follows that $0 \le f(x) \le 1 \ \forall x \in X$. Now to show that f is $\mathcal{J}$-$\mathcal{J}^*$ continuous where $\mathcal{J}^*$ denotes the $\mathcal{J}$-relative topology for $[0,1]$.

Now all the interval of the form $[0,a)$ and $(a,1]$ for $0<a<1$. form an open sub-base for $[0,1]$. Hence in order to prove that f is continuous it will be sufficient if we prove that the inverse image under f of both $[0,a)$ and $(a,1]$ is $\mathcal{J}$-open $\forall\ 0<a<1$.

Now $\quad f^{-1}([0,a)) = \{x \in X : f(x) \in [0,a)\}$
$= \{x \in X : f(x) < a\}$ — ①

Now $\quad f(x) < a \Rightarrow x \in G_d$ for some $d<a$. — ②

Hence from ① and ②, we conclude that

$$f^{-1}([0,a)) = \cup\{G_d : d<a\}.$$

Since each G_d is $\mathcal{J}$-open the arbitrary union is also $\mathcal{J}$-open. and hence $f^{-1}([0,a))$ is also $\mathcal{J}$-open

Again $\quad f^{-1}((a,1]) = \{x \in X : g(x) \in [a,1]\}$
$= \{x \in X : a < g(x)\}$ — ③

Now $\quad a < f(x) \Rightarrow x \notin \bar{G}_d$ for some $d>a$
$\Rightarrow x \in (\bar{G}_d)^c$ for some $d>a$.

Hence from ③ and ④ we conclude that

$$f^{-1}((a,1]) = \cup\{(\bar{G}_d)^c : d>a\}$$

Since each $(\bar{G}_d)^c$ is $\mathcal{J}$-open, the arbitrary union is also $\mathcal{J}$-open ~~to~~ and hence $f^{-1}((a,1))$ be $\mathcal{J}$-open.

Hence $f : X \to [0,1]$ is $\mathcal{J}$-$\mathcal{J}'$ continuous mapping such that

$$f(x) = \begin{cases} 0 & \text{if } x \in F_1 \\ 1 & \text{if } x \in F_2 \end{cases}$$